

HOLOMORPHIC FUNCTIONS UNBOUNDED ON CURVES OF FINITE LENGTH

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Abstract Given a pseudoconvex domain $D \subset \mathbb{C}^N$, $N \geq 2$, we prove that there is a holomorphic function f on D such that the lengths of paths $p: [0, 1] \rightarrow D$ along which $\Re f$ is bounded above, with $p(0)$ fixed, grow arbitrarily fast as $p(1) \rightarrow bD$. A consequence is the existence of a complete closed complex hypersurface $M \subset D$ such that the lengths of paths $p: [0, 1] \rightarrow M$, with $p(0)$ fixed, grow arbitrarily fast as $p(1) \rightarrow bD$.

1. Introduction and the main results

Denote by Δ the open unit disc in \mathbb{C} and by \mathbb{B}_N the open unit ball of \mathbb{C}^N , $N \geq 2$. In [G] it was proved that there is a closed complex hypersurface M in \mathbb{B}_N which is complete, that is, every path $p: [0, 1) \rightarrow M$ such that $|p(t)| \rightarrow 1$ as $t \rightarrow 1$, has infinite length. This was a consequence of the main result of [G], a construction of a holomorphic function on \mathbb{B}_N whose real part is unbounded on every path of finite length that ends on $b\mathbb{B}_N$.

Recall that a domain $D \subset \mathbb{C}^N$, $N \geq 2$, is *pseudoconvex* if it has a continuous plurisubharmonic exhaustion function. This happens if and only if D is holomorphically convex and if and only if D is a domain of holomorphy [H]. Every convex domain is pseudoconvex. In the present paper we show that given a pseudoconvex domain D in \mathbb{C}^N , $N \geq 2$, there is a holomorphic function f on D such that the lengths of paths $p: [0, 1] \rightarrow D$ along which the real part of f is bounded above, grow arbitrarily rapidly if $p(0)$ is fixed and $p(1)$ tends to bD . Our main result is the following

THEOREM 1.1 *Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain and let D_n , $n \in \mathbb{N}$ be an exhaustion of D by relatively compact open sets*

$$D_1 \subset\subset D_2 \subset\subset \cdots \subset D, \quad \bigcup_{n=1}^{\infty} D_n = D.$$

Let A_n , $n \in \mathbb{N}$, be an increasing sequence of positive numbers converging to $+\infty$. There is a function f , holomorphic on D , with the following property:

Given $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \rightarrow D$ is a path such that

$$(i) \quad \Re[f(p(t))] \leq L \quad (0 \leq t \leq 1)$$

$$(ii) \quad p(0) \in D_1, \quad p(1) \in D \setminus D_n$$

then the length of p exceeds A_n .

So, in particular, for every $L < \infty$, the boundary bD is infinitely far away for a traveller travelling within a sublevel set $\{z \in D: \Re(f(z)) < L\}$ of the real part of f :

COROLLARY 1.1 *Given a pseudoconvex domain $D \subset \mathbb{C}^N$, $N \geq 2$, there is a holomorphic function on D whose real part is unbounded above on every path $p: [0, 1) \rightarrow D$, $p(1) \in bD$, of finite length.*

It is perhaps worth mentioning that for any holomorphic function f on \mathbb{B}_N there are paths $p: [0, 1] \rightarrow \mathbb{B}_N$, $p([0, 1)) \subset \mathbb{B}_N$, $p(1) \in b\mathbb{B}_N$ along which f is constant [GS].

Let $M \subset D$ be a closed complex hypersurface, that is, a closed complex submanifold of D of complex codimension one. A path $p: [0, 1) \rightarrow M$ is called *divergent* if $p(t)$ leaves every compact subset of M as $t \rightarrow 1$. M is called *complete* if every divergent path $p: [0, 1) \rightarrow M$ has infinite length.

Let f be as in Corollary 1.1. By Sard's theorem one can choose $c \in \mathbb{C}$ such that

$$M = \{z \in D: f(z) = c\}$$

is a complex manifold. By the properties of f , M is a complete hypersurface. So we have

COROLLARY 1.2 *Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain. Then D contains a complete closed complex hypersurface.*

In the special case when $D = \mathbb{B}_N$, Corollary 1.1 and Corollary 1.2 were proved in [G]. In [AL] Corollary 1.2 was proved for convex domains in \mathbb{C}^2 . Note that Theorem 1.1 implies a stronger result - given an exhaustion D_j , $j \in \mathbb{N}$, of a pseudoconvex domain D as in Theorem 1.1, there is a complete closed complex hypersurface M in D such that along paths, $M \setminus D_j$ becomes arbitrarily far away as $j \rightarrow \infty$:

COROLLARY 1.3 *Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain, and let D_j and A_j , $j \in \mathbb{N}$, be as in Theorem 1.1. There is a complete closed complex hypersurface M in D meeting D_1 with the following property: There is some $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then the length of every path $p: [0, 1] \rightarrow M$, such that $p(0) \in D_1$ and $p(1) \in M \setminus D_n$, exceeds A_n .*

2. The main lemma. Reduction to the case $D = \mathbb{C}^N$

We assume that $N \geq 2$ and write \mathbb{B} for \mathbb{B}_N . We shall use *spherical shells*. If J is an interval contained in $(0, \infty)$ we shall write $\text{Sh}(J) = \{x \in \mathbb{C}^N: |x| \in J\}$. So, if $J = [\alpha, \beta]$ then $\text{Sh}(J) = \{x \in \mathbb{C}^N: \alpha \leq |x| \leq \beta\} = \beta\overline{\mathbb{B}} \setminus \alpha\overline{\mathbb{B}}$. Here is our main lemma:

LEMMA 2.1 *Let $J = (r, R)$ where $0 < r < R < \infty$ and let $A < \infty$. There is a set $E \subset \text{Sh}(J)$ such that*

- (i) *the length of every path $p: [0, 1] \rightarrow \text{Sh}(\overline{J}) \setminus E$ such that $|p(0)| = r$, $|p(1)| = R$, exceeds A*
- (ii) *given $\varepsilon > 0$ and $L < \infty$ there is a polynomial Φ on \mathbb{C}^N such that $|\Phi| < \varepsilon$ on $r\overline{\mathbb{B}}$ and $\Re\Phi > L$ on E .*

We will prove Lemma 2.1 in the following sections. To prove Theorem 1.1 we need the following consequence of Lemma 2.1.

LEMMA 2.2 *Let $0 < r_1 < R_1 < r_2 < R_2 < \dots$, $r_n \rightarrow \infty$ as $n \rightarrow \infty$, and let B_n be an increasing sequence of positive numbers converging to $+\infty$. There is a holomorphic function g on \mathbb{C}^N , $N \geq 2$, such that for any $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \rightarrow \mathbb{C}^N$ is a path such that*

$$(i) \Re[g(p(t))] \leq L \quad (0 \leq t \leq 1)$$

$$(ii) |p(0)| \leq r_n, \quad |p(1)| \geq R_n$$

then the length of p exceeds B_n .

Proof. Let $0 < r_1 < R_1 < r_2 < R_2 < \dots$, $r_n \rightarrow +\infty$ as $n \rightarrow \infty$ and let B_n be an increasing sequence of positive numbers, converging to $+\infty$. By Lemma 2.1 there is, for each n , a set $E_n \subset \text{Sh}((r_n, R_n))$ such that

- the length of every path $p: [0, 1] \rightarrow \text{Sh}([r_n, R_n]) \setminus E_n$ such that $|p(0)| = r_n$, $|p(1)| = R_n$, exceeds B_n
- given $\varepsilon > 0$ and $L < \infty$ there is a polynomial Ψ on \mathbb{C}^N such that $|\Psi| < \varepsilon$ on $r_n \overline{\mathbb{B}}$ and $\Re \Psi > L$ on E_n .

Let L_n be an increasing sequence converging to $+\infty$. Suppose for a moment that we have a sequence of polynomials Φ_n such that

$$(a) \Re \Phi_n > L_n + 1 \text{ on } E_n$$

$$(b) |\Phi_{n+1} - \Phi_n| < 1/2^n \text{ on } R_n \overline{\mathbb{B}}.$$

By (b) the sequence Φ_n converges uniformly on compacta on \mathbb{C}^N , denote by g its limit. So g is holomorphic on \mathbb{C}^N . On E_n we have $\Re g = \Re[\Phi_n + \sum_{j=n}^{\infty} (\Phi_{j+1} - \Phi_j)] \geq \Re \Phi_n - \sum_{j=n}^{\infty} 2^{-j} \geq L_n + 1 - 1 = L_n$.

Let $L < \infty$. There is an n_0 such that $L < L_n$ for all $n \geq n_0$. Suppose that a path p satisfies (i). Then there are α, β , $0 < \alpha < \beta < 1$ such that $p((\alpha, \beta) \subset \text{Sh}((r_n, R_n))$ and $p(\alpha) = r_n$, $p(\beta) = R_n$. If p satisfies also (i) then, since $\Re g \geq L_n$ on E_n it follows that $p|_{[\alpha, \beta]}$ is a map to $\text{Sh}([r_n, R_n]) \setminus E_n$ so by the preceding discussion the length of $p|_{[\alpha, \beta]}$ exceeds B_n and consequently the length of p exceeds B_n .

We construct the sequence Φ_n by induction. Pick a polynomial Φ_1 such that $\Re \Phi_1 > L_1 + 1$ on E_1 . Suppose that we have constructed Φ_n . There is a constant $C < \infty$ such that $\Re(\Phi_n + C) \geq L_{n+1} + 1$ on E_{n+1} . By the preceding discussion there is a polynomial Ψ such that $|\Psi| < 1/2^n$ on $R_n \overline{\mathbb{B}}$ and $\Re \Psi > C$. Then $\Phi_{n+1} = \Phi_n + \Psi$ satisfies (b) and (a) with n replaced by $n + 1$. This completes the proof.

The same proof gives an analogous result for the ball which we will not need in the sequel:

COROLLARY 2.1 *Let $0 < r_1 < R_1 < r_2 < R_2 < \dots$, $r_n \rightarrow 1$ as $n \rightarrow \infty$, and let A_n be an increasing sequence of positive numbers converging to ∞ . There is a holomorphic function g on \mathbb{B} , such that for any $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \rightarrow \overline{\mathbb{B}}$ is a path such that*

$$(i) \Re[g(p(t))] \leq L \quad (0 \leq t \leq 1)$$

$$(ii) |p(0)| \leq r_n, \quad |p(1)| \geq R_n$$

then the length of p exceeds A_n .

Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain. Then D is a Stein manifold so there is a proper holomorphic embedding $F: D \rightarrow \mathbb{C}^{2N+1}$ [H, Th.5.3.9].

To prove Theorem 1.1. we first prove the following consequence of Lemma 2.2.

LEMMA 2.3 *Let $0 < r_1 < R_1 < r_2 < R_2 < \dots$, $r_n \rightarrow \infty$ as $n \rightarrow \infty$, and let A_n be an increasing sequence of positive numbers converging to ∞ . There is a holomorphic function f on D such that for any $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \rightarrow D$ is a path such that*

- (i) $\Re[f(p(t))] \leq L$ ($0 \leq t \leq 1$)
- (ii) $|F(p(0))| \leq r_n$, $|F(p(1))| \geq R_n$

then the length of p exceeds A_n .

Note that Lemma 2.3 implies a more precise version of Theorem 1.1, yet for a specific exhaustion $D_n = \{z \in D: |F(z)| < R_n\}$, $n \in \mathbb{N}$.

Proof of Lemma 2.3 Let $K_n = \{z \in D: r_n \leq |F(z)| \leq R_n\}$. Let $p: [0, 1] \rightarrow K_n$ be a path. Then $q = F \circ p: [0, 1] \rightarrow \mathbb{C}^{2N+1}$ is a path whose length equals

$$\begin{aligned} \text{length}(q) &= \int_0^1 \left| (DF)(p(t)) \left(\frac{dp}{dt}(t) \right) \right| dt \\ &\leq \max_{w \in K_n} \|(DF)(w)\| \int_0^1 \left| \frac{dp}{dt}(t) \right| dt \\ &= \max_{w \in K_n} \|(DF)(w)\| \cdot \text{length}(p). \end{aligned}$$

The map F is holomorphic and K_n is compact so

$$\max_{w \in K_n} \|(DF)(w)\| < \infty.$$

Let A_n , $n \in \mathbb{N}$, be an increasing sequence converging to $+\infty$. Choose an increasing sequence B_n converging to $+\infty$ such that

$$A_n \cdot \max_{w \in K_n} \|(DF)(w)\| \leq B_n \quad (n \in \mathbb{N}). \quad (2.1)$$

Let g be an entire function on \mathbb{C}^{2N+1} given by Lemma 2.2 and let $f = g \circ F$. Let $L < \infty$. By Lemma 2.2 there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $s: [0, 1] \rightarrow \mathbb{C}^{2N+1}$ is a path such that

- (i) $\Re[g(s(t))] \leq L$ ($0 \leq t \leq 1$)
- (ii) $|s(0)| \leq r_n$, $|s(1)| \geq R_n$

then the length of s exceeds B_n .

Now, let $n \geq n_0$ and let $p: [0, 1] \rightarrow D$ be a path such that $|F(p(0))| \leq r_n$, $|F(p(1))| \geq R_n$ and $\Re[f(p(t))] \leq L$ ($0 \leq t \leq 1$), that is

$$\Re[g(s(t))] \leq L \quad (0 \leq t \leq 1) \quad (2.2)$$

where $s = F \circ p$. There is a segment $[\alpha, \beta] \subset [0, 1]$ such that $p|_{[\alpha, \beta]}$ maps $[\alpha, \beta]$ to K_n and $|F(p(\alpha))| = r_n$, $|F(p(\beta))| = R_n$, that is, $|s(\alpha)| = r_n$, $|s(\beta)| = R_n$. By (2.2) Lemma 2.2 implies that

$$\text{length}(s|_{[\alpha, \beta]}) \geq B_n.$$

By (2.1) it follows that

$$\begin{aligned}
\text{length}(p|[\alpha, \beta]) &\geq \frac{\text{length}((F \circ p)|[\alpha, \beta])}{\max_{w \in K_n} \|(DF)(w)\|} \\
&= \frac{\text{length}(s|[\alpha, \beta])}{\max_{w \in K_n} \|(DF)(w)\|} \\
&\geq \frac{B_n}{\max_{w \in K_n} \|(DF)(w)\|} \\
&\geq A_n
\end{aligned}$$

Thus, the length of p exceeds A_n . This completes the proof of Lemma 2.3 provided that Lemma 2.1 has been proved.

Proof of Theorem 1.1. Let D_j , $j \in \mathbb{N}$, be as in Theorem 1.1 and let $w \in D_1$. Since $F: D \rightarrow \mathbb{C}^{2N+1}$ is a proper map there are $m_0 \in \mathbb{N}$ and a strictly increasing sequence $R_n \nearrow \infty$ such that if $\Delta_n = \{z \in D: |F(z)| < R_n\}$, $n \in \mathbb{N}$ then $D_1 \subset \Delta_1$ and $\Delta_n \subset D_n$ ($n \geq m_0$). By Lemma 2.3 there is a holomorphic function f on D such that for any $L < \infty$ there is an $n_0 \in \mathbb{N}$, $n_0 \geq m_0$, such that if $n \geq n_0$ and if $p: [0, 1] \rightarrow D$ is a path such that

$$\Re[f(p(t))] \leq L \quad (0 \leq t \leq 1), \quad (2.3)$$

$p(0) \in \Delta_1$ and $p(1) \in D \setminus \Delta_n$ then the length of p exceeds A_n . Since $D_1 \subset \Delta_1$ and $\Delta_n \subset D_n$ ($n \geq n_0$) the same holds for any path $p: [0, 1] \rightarrow D$ which satisfies (2.3) and $p(0) \in D_1$, $p(1) \in D \setminus D_n$. This completes the proof of Theorem 1.1.

Remark Note that we used only the fact that $F: D \rightarrow \mathbb{C}^{2N+1}$ is a proper holomorphic map. We did not need the fact that it is an injective immersion.

It remains to prove Lemma 2.1.

3. Proof of Lemma 2.1, Part 1

If I_1, I_2 are two intervals contained in $(0, \infty)$ then we shall write $I_1 < I_2$ provided that $I_1 \cap I_2 = \emptyset$ and provided that I_2 is to the right of I_1 , that is, if $x_1 < x_2$ for every $x_1 \in I_1$, $x_2 \in I_2$. If $\mathcal{V} \subset b\mathbb{B}$ is an open set and $J \subset (0, \infty)$ is an interval then we call the set

$$K(\mathcal{V}, J) = \{tz: z \in \mathcal{V}, t \in J\}$$

a *spherical box*. Clearly

$$K(\mathcal{V}, J) = \{x \in \text{Sh}(J): \frac{x}{|x|} \in \mathcal{V}\}.$$

A set of the form $\{x \in b\mathbb{B}: |x - x_0| < \eta\}$ where $x_0 \in b\mathbb{B}$ is called a *ball in $b\mathbb{B}$ of radius η* . We show that Lemma 2.1 follows from

LEMMA 3.1. *There is a $\rho > 0$ with the following property :*

For every ball $\mathcal{V} \subset b\mathbb{B}$ of radius ρ , for every $A < \infty$ and for every open interval $J = (\alpha, \beta)$ where $1/2 < \alpha < \beta < 1$ there is a set $E \subset K(\mathcal{V}, J)$ such that

(i) the length of every path $p: [0, 1] \rightarrow K(\mathcal{V}, \overline{J}) \setminus E$ such that $|p(0)| = \alpha$, $|p(1)| = \beta$, exceeds A

(ii) given $\varepsilon > 0$ and $L < \infty$ there is a polynomial Φ on \mathbb{C}^N such that $|\Phi| < \varepsilon$ on $\alpha\overline{\mathbb{B}}$ and $\Re\Phi > L$ on E .

One can view Lemma 3.1 as a local version of Lemma 2.1. Note, however, that ρ does not depend on J .

Proof of Lemma 2.1, assuming Lemma 3.1. Let $J = (r, R)$ where $0 < r < R < \infty$ and let $A < \infty$. It is easy to see that it is enough to prove Lemma 2.1 in the case when R is close to r . Hence, with no loss of generality assume that $1/2 < r < R < 1$.

Let $\rho > 0$ be as in Lemma 3.1 and put $\eta = \rho/4$. Choose points $w_1, w_2, \dots, w_M \in b\mathbb{B}$ such that the balls

$$\mathcal{V}_i = \{w \in b\mathbb{B}: |w - w_i| < 2\eta\}, \quad 1 \leq i \leq M,$$

cover $b\mathbb{B}$ and then let

$$\mathcal{W}_i = \{w \in b\mathbb{B}: |w - w_i| < 4\eta\}, \quad 1 \leq i \leq M.$$

Then every ball $\mathcal{V} \subset b\mathbb{B}$ of radius 2η is contained in at least one of \mathcal{W}_j , $1 \leq j \leq M$. Thus, if $p: [0, 1] \rightarrow b\mathbb{B}$ is a path then either $p([0, 1])$ is contained in \mathcal{W}_i for some i , $1 \leq i \leq M$ or else the length of p exceeds 2η . If $p: [0, 1] \rightarrow \text{Sh}(\overline{J}) \subset \text{Sh}((1/2, 1))$ is a path then looking at the radial projections $\pi_R(z) = z/|z|$ we conclude that either $(\pi_R \circ p)([0, 1])$ is contained in \mathcal{W}_i for some i , $1 \leq i \leq M$, or else the length of $\pi_R \circ p$ exceeds 2η which implies that the length of p exceeds η . Thus we have

$$\left. \begin{array}{l} \text{Let } p: [0, 1] \rightarrow \text{Sh}(J) \text{ be a path. Then either there is some } j, 1 \leq j \leq M \\ \text{such that } p([0, 1]) \subset K(\mathcal{W}_j, J) \text{ or else the length of } p \text{ exceeds } \eta. \end{array} \right\} \quad (3.1)$$

Choose $\ell \in \mathbb{N}$ so large that

$$\ell\eta > A. \quad (3.2)$$

Divide the interval J into ℓ pieces $J_1 = (r, r + (R - r)/\ell), \dots, J_\nu = ((r + (\nu - 1)(R - r))/\ell, r + \nu(R - r)/\ell), \dots, J_\ell = (r + (\ell - 1)(R - r)/\ell, R)$. Clearly

$$(0, r) < J_1 < J_2 < \dots < J_\ell < (R, \infty)$$

and the length $|J_k|$ of each J_k , $1 \leq k \leq \ell$ equals $(R - r)/\ell$. For each k , $1 \leq k \leq \ell$, divide J_k into M equally long pieces J_{ks} , $1 \leq s \leq M$, so that J_{ks} are pairwise disjoint open intervals contained in J_k such that

$$J_{k1} < J_{k2} < \dots < J_{kM} \quad \text{and} \quad |J_{ks}| = |J_k|/M = (R - r)/(\ell M) \quad (1 \leq s \leq M).$$

For each k, s , $1 \leq k \leq \ell, 1 \leq s \leq M$, we apply Lemma 3.1 for the ball $\mathcal{V} = \mathcal{W}_s$ and the interval J_{ks} to get a set $E_{ks} \subset K(\mathcal{W}_s, J_{ks})$ such that

$$\left. \begin{array}{l} \text{if } J_{ks} = (\alpha_{ks}, \beta_{ks}) \text{ then the length of every path } p: [0, 1] \rightarrow K(\mathcal{W}_s, \overline{J_{ks}}) \setminus E_{ks} \\ \text{such that } |p(0)| = \alpha_{ks}, |p(1)| = \beta_{ks}, \text{ exceeds } A, \end{array} \right\} \quad (3.3)$$

and such that

$$\left. \begin{array}{l} \text{given } \varepsilon > 0 \text{ and } L < \infty \text{ there is a polynomial } \Phi \\ \text{on } \mathbb{C}^N \text{ such that } |\Phi| < \varepsilon \text{ on } \alpha_{ks} \overline{\mathbb{B}} \text{ and } \Re \Phi > L \text{ on } E_{ks}. \end{array} \right\} \quad (3.4)$$

Put

$$E = \bigcup_{s=1}^M \bigcup_{k=1}^{\ell} E_{ks}.$$

We show that E has the required properties.

Clearly $E \subset \text{Sh}(J)$. Let $p: [0, 1] \rightarrow \text{Sh}(J) \setminus E$ be a path such that $|p(0)| = r$, $|p(1)| = R$. Let $J_k = (\alpha_k, \beta_k)$ ($1 \leq k \leq \ell$). Since $|p(0)| = r$, $|p(1)| = R$ it follows that for each k , $1 \leq k \leq \ell$, there are γ_k, γ'_k such that

$$0 \leq \gamma_1 < \gamma'_1 < \gamma_2 < \gamma'_2 < \cdots < \gamma_\ell < \gamma'_\ell \leq 1,$$

such that for each k , p maps $[\gamma_k, \gamma'_k]$ to $\text{Sh}(\overline{J_k})$ and (γ_k, γ'_k) to $\text{Sh}(J_k)$ and satisfies $|p(\gamma_k)| = \alpha_k$, $|p(\gamma'_k)| = \beta_k$.

Fix k , $1 \leq k \leq \ell$. By (3.1) there are two possibilities. Either there is some s , $1 \leq s \leq M$, such that $p([\gamma_k, \gamma'_k]) \subset K(\mathcal{W}_s, \overline{J_k})$ or else the length of $p([\gamma_k, \gamma'_k])$ exceeds η . Assume that the first happens. Write $J_{ks} = (\alpha_{ks}, \beta_{ks})$. Since $p([\gamma_k, \gamma'_k]) \subset K(\mathcal{W}_s, \overline{J_k})$ and $|p(\gamma_k)| = \alpha_k$, $|p(\gamma'_k)| = \beta_k$ it follows that there are $\gamma_{ks}, \gamma'_{ks}$ such that $\gamma_k < \gamma_{ks} < \gamma'_{ks} < \gamma'_k$, and such that $|p(\gamma_{ks})| = \alpha_{ks}$, $|p(\gamma'_{ks})| = \beta_{ks}$ and $p([\gamma_{ks}, \gamma'_{ks}]) \subset K(\mathcal{W}_s, \overline{J_{ks}})$. Clearly p maps $[\gamma_{ks}, \gamma'_{ks}]$ into $K(\mathcal{W}_s, \overline{J_{ks}}) \setminus E = K(\mathcal{W}_s, \overline{J_{ks}}) \setminus E_{ks}$ so by (3.3) the length of $p([\gamma_{ks}, \gamma'_{ks}])$ exceeds A and so the length of $p([\gamma_k, \gamma'_k])$ exceeds A . This shows that for each k , $1 \leq k \leq \ell$, the length of $p([\gamma_k, \gamma'_k])$ exceeds $\min\{\eta, A\}$ and hence by (3.2) the length of p exceeds A . This shows that E satisfies (i) in Lemma 2.1.

It remains to show (ii) in Lemma 2.1. To this end, rename the intervals J_{ks} , $1 \leq k \leq \ell$, $1 \leq s \leq M$, into $I_1, I_2, \dots, I_{\ell M}$ and the sets E_{ks} , $1 \leq k \leq \ell$, $1 \leq s \leq M$, into $E_1, E_2, \dots, E_{\ell M}$ in such a way that

$$(0, r) < I_1 < I_2 < \cdots < I_{\ell M} < (R, \infty)$$

and that $E_j \subset \text{Sh}(I_j)$ ($1 \leq j \leq \ell M$). There are μ_j , $1 \leq j \leq \ell M + 1$, such that $\mu_1 = r$, $\mu_{\ell M + 1} = R$, and such that $I_j = (\mu_j, \mu_{j+1})$ ($1 \leq j \leq \ell M$). Recall that by the properties of E_j ,

$$\left. \begin{array}{l} \text{for each } j, 1 \leq j \leq \ell M, \text{ and for each } \varepsilon > 0 \text{ and } L < \infty \text{ there} \\ \text{is a polynomial } \Psi \text{ such that } |\Psi| < \varepsilon \text{ on } \mu_j \overline{\mathbb{B}} \text{ and } \Re \Psi > L \text{ on } E_j. \end{array} \right\} \quad (3.5)$$

Let $L < \infty$ and let $\varepsilon > 0$. Let Φ_1 be a polynomial such that

$$|\Phi_1| < \frac{\varepsilon}{\ell M} \quad \text{on } \mu_1 \overline{\mathbb{B}} = r \overline{B} \quad \text{and} \quad \Re \Phi_1 > L + \varepsilon \quad \text{on } E_1$$

which is possible by (3.5). We construct polynomials Φ_j , $2 \leq j \leq \ell M$, such that for each j , $2 \leq j \leq \ell M$,

$$|\Phi_j - \Phi_{j-1}| < \frac{\varepsilon}{M\ell} \quad \text{on } \mu_j \overline{B} \quad \text{and} \quad \Re \Phi_j > L + \varepsilon \quad \text{on } E_j \quad (3.6)$$

and then put $\Phi = \Phi_{M\ell}$. We show that Φ has the required properties. On $r\overline{B} = \mu_1 \overline{B}$ we have $|\Phi| \leq |\Phi_1| + |\Phi_2 - \Phi_1| + \dots + |\Phi_{M\ell} - \Phi_{M\ell-1}| < M\ell \cdot \varepsilon / (M\ell) = \varepsilon$. Fix j , $1 \leq j \leq \ell M$. On E_j we have $\Re \Phi = \Re [\Phi_j + (\Phi_{j+1} - \Phi_j) + \dots + (\Phi_{M\ell} - \Phi_{M\ell-1})] \geq \Re \Phi_j - (M\ell - 1)\varepsilon / M\ell \geq L + \varepsilon - \varepsilon$. Thus, on $E = \cup_{j=1}^{M\ell} E_j$ we have $\Re \Phi > L$.

To find $\Phi_2, \dots, \Phi_{\ell M}$ satisfying (3.6) we use (3.5): Suppose that we have constructed Φ_j where $1 \leq j \leq \ell M - 1$. There is a constant $C < \infty$ such that $\Re \Phi_j + C \geq L + \varepsilon$ on E_{j+1} . By (3.5) there is a polynomial Ψ such that $|\Psi| < \varepsilon / (M\ell)$ on $\mu_{j+1} \overline{B}$ and $\Re \Psi > C$ on E_{j+1} . Then $\Phi_{j+1} = \Phi_j + \Psi$ has all the required properties. This completes the proof.

4. Proof of Lemma 3.1, Part 1.

Write $M = 2N$ and identify \mathbb{C}^N with \mathbb{R}^M by identifying $(p_1 + iq_1, \dots, p_N + iq_N) \in \mathbb{C}^N$ with $(p_1, q_1, \dots, p_N, q_N) \in \mathbb{R}^M$. Let U_0, U_1, U be small open balls in \mathbb{R}^{M-1} centered at the origin, such that

$$U \subset \subset U_1 \subset \subset U_0.$$

Write $W_0 = U_0 \times (0, \infty)$. This is an open half tube in $\mathbb{R}^M = \mathbb{C}^N$. Similarly, write $W_1 = U_1 \times (0, \infty)$, $W = U \times (0, \infty)$. Given an interval $J \subset (1/2, \infty)$ we shall write

$$W_0(J) = \text{Sh}(J) \cap W_0, \quad W_1(J) = \text{Sh}(J) \cap W_1, \quad W(J) = \text{Sh}(J) \cap W.$$

We assume that the ball U_0 is so small that for each r , $1/2 < r < 1$, the surface $W_0 \cap b(r\overline{B})$ can be written as the graph of the function

$$\psi_r(x_1, \dots, x_{M-1}) = \left(r^2 - \sum_{i=1}^{M-1} x_i^2 \right)^{1/2}$$

defined on U_0 , that is,

$$W_0 \cap b(r\overline{B}) = \{(x_1, \dots, x_{M-1}, \psi_r(x_1, \dots, x_{M-1})) : (x_1, \dots, x_{M-1}) \in U_0\}.$$

We now turn to the proof of Lemma 3.1. By rotation it is enough to prove that there is one ball $\mathcal{V} \subset b\overline{B}$ of radius $\rho > 0$ with the properties in Lemma 3.1. It is easy to see that to prove this it is enough to prove that there is a ball U as above such that for every $A < \infty$ and for every segment $J = (\alpha, \beta]$, $1/2 < \alpha < \beta < 1$, there is a set $E \subset W(J)$ such that

$$\left. \begin{array}{l} (i) \text{ the length of every path } p: [0, 1] \rightarrow W(\overline{J}) \setminus E \text{ such that} \\ |p(0)| = \alpha, |p(1)| = \beta, \text{ exceeds } A \\ (ii) \text{ given } \varepsilon > 0 \text{ and } L < \infty \text{ there is a polynomial } \Phi \text{ on } \mathbb{C}^N \text{ such that} \\ |\Phi| < \varepsilon \text{ on } \alpha\overline{B} \text{ and } \Re \Phi > L \text{ on } E. \end{array} \right\} \quad (4.1)$$

We now use some ideas from [GS] and [G]. Let us describe briefly how the set E will look like. We will construct finitely many intervals J_j , $1 \leq j \leq n$, $(-\infty, 1/2) < J_1 < \dots < J_n < [1, \infty)$. For each j , $1 \leq j \leq n$, we will construct a convex polyhedral surface $C_j \subset \text{Sh}(J_j)$ whose facets will be simplices which is such that $W \setminus C_j$ has two components. From each C_j we shall remove a tiny neighbourhood \mathcal{U}_j of the skeleton of C_j and what remains intersect with W to get the set E_j . The set E will be the union of E_j , $1 \leq j \leq n$. A path $p: [0, 1] \rightarrow W([\alpha, \beta]) \setminus E$, such that $|p(0)| = \alpha$, $|p(1)| = \beta$ will have to pass through each C_j , and will have to meet C_j in the neighbourhood \mathcal{U}_j of $\text{Skel}(C_j)$. We shall show that given $A < \infty$, $n \in \mathbb{N}$, the intervals J_j , the convex surfaces C_j and \mathcal{U}_j , $1 \leq j \leq n$ can be chosen in such a way that (i) in (4.1) will hold. The fact that C_j are convex and contained in disjoint spherical shells will enable us to satisfy (ii) in (4.1).

Begin with a tessellation \mathcal{T} of \mathbb{R}^{M-1} into simplices which is periodic with respect to a lattice

$$\Lambda = \left\{ \sum_{i=1}^{M-1} n_i e_i : n_i \in \mathbb{Z}, 1 \leq i \leq M-1 \right\}$$

where $\{e_1, \dots, e_{M-1}\}$ is a basis of \mathbb{R}^{M-1} , that is $S + e \in \mathcal{T}$ for every simplex $S \in \mathcal{T}$ and for every $e \in \Lambda$. What remains of \mathbb{R}^{M-1} after we remove the interiors of all simplices in \mathcal{T} we call the *skeleton* of \mathcal{T} and denote by $\text{Skel}(\mathcal{T})$. More generally, we shall use the tessellations

$$\tau(\mathcal{T} + z) = \{\tau(S + z) : S \in \mathcal{T}\}$$

where $z \in \mathbb{R}^{M-1}$ and $\tau > 0$ and define $\text{Skel}(\tau(\mathcal{T} + z))$ in the same way.

We now show how to construct the polyhedral surfaces mentioned above.

Fix U_0, U_1 and U as above. Fix $z \in \mathbb{R}^{M-1}$ and let $\tau > 0$ be very small. Fix r , $1/2 < r < 1$. To get the vertices of our polyhedral surface we shall "lift" the vertices of each simplex $S \in \tau(\mathcal{T} + z)$ contained in U_0 to $b(r\mathbb{B})$ in the sense that if $v_1, \dots, v_M \in \mathbb{R}^{M-1}$ are the vertices of S then $(v_i, \psi_r(v_i))$, $1 \leq i \leq M$ are the vertices of the simplex that we denote by $\Psi_r(S)$. The union of these simplices $\Psi_r(S)$ for all $S \in \tau(\mathcal{T} + z)$ contained in U_0 we denote by $\Gamma(r, \tau, z)$. This is a polyhedral surface. It is the graph of the piecewise linear function $\varphi_{r, \tau, z}$ defined on the union of all simplices S as above, where, on each such simplex with vertices v_1, \dots, v_M we have

$$\varphi_{r, \tau, z} \left(\sum_{i=1}^M \lambda_i v_i \right) = \sum_{i=1}^M \lambda_i \psi_r(v_i) \quad (0 \leq \lambda_i \leq 1, 1 \leq i \leq M, \sum_{i=1}^M \lambda_i = 1).$$

We will show later that the tessellation \mathcal{T} can be chosen in such a way that if U_0 is chosen small enough then for each r , $1/2 < r < 1$, and each z , the surface $\Gamma(r, \tau, z)$ will be *convex* in the sense that given a simplex $\Psi_r(S)$ where $S \in \tau(\mathcal{T} + z)$ is contained in U_0 , the intersection of the hyperplane H containing $\Psi_r(S)$ with $\Gamma(r, \tau, z)$ is precisely $\Psi_r(S)$, that is, all of $\Gamma(r, \tau, z)$ except $\Psi_r(S)$ is contained in the open halfspace bounded by H which contains the origin.

Let d be the length of the longest edge of the simplices in \mathcal{T} . Then τd is the length of the longest edge of the simplices in $\tau(\mathcal{T} + z)$ for any $\tau > 0$ and any $z \in \mathbb{R}^{N-1}$. There is a constant $\nu > 0$ depending on U_0 such that for each r , $1/2 < r < 1$, the length of the

longest edge of a simplex building $\Gamma(r, \tau, z)$ does not exceed $\gamma = \nu\tau d$. Thus the vertices of each such simplex are contained in a spherical cap $\{x \in b(r\mathbb{B}): |x - x_0| < \gamma\}$ for some $x_0 \in b(r\mathbb{B})$ and consequently the simplex is contained in the convex hull of this spherical cap. If $1/2 < r < 1$ then it is easy to see that this convex hull misses $(r - 2\gamma^2)\overline{\mathbb{B}}$. It follows that there is a constant $\omega = 2\nu^2 d^2$ such that

$$\left. \begin{array}{l} \text{for each } r, \ 1/2 < r < 1, \text{ each } z \in \mathbb{R}^{M-1} \text{ and each} \\ \tau > 0 \text{ we have } \Gamma(r, \tau, z) \subset \text{Sh}((r - \omega\tau^2, r]). \end{array} \right\} \quad (4.2)$$

It is a simple geometric fact that there is a $\delta > 0$ such that

$$\left. \begin{array}{l} \text{for every } r, \ 1/2 < r < 1, \text{ and for every } s, \ r - \delta < s < r, \text{ every line} \\ \text{that meets } W((s, r]) \text{ and misses } W_1 \cap b(s\mathbb{B}), \text{ misses } s\overline{\mathbb{B}}. \end{array} \right\} \quad (4.3)$$

There is a $\tau_0 > 0$ such that $\omega\tau_0^2 < \delta$ where δ satisfies (4.3) and which is so small that for every τ , $0 < \tau < \tau_0$, and for every $z \in \mathbb{R}^{M-1}$ the union of all simplices in $\tau(\mathcal{T} + z)$ contained in U_0 , contains U_1 . Let $0 < \tau < \tau_0$ and let $z \in \mathbb{R}^{M-1}$. Then $\Gamma(r, \tau, z) \cap W_1$ is a graph over U_1 which is contained in $W_1((r - \omega\tau^2, r])$.

We now show that each hyperplane H meeting $\Gamma(r, \tau, z) \cap W$ and tangent to $\Gamma(r, \tau, z)$ misses $(r - \omega\tau^2)\overline{\mathbb{B}}$. This is easy to see. By the convexity of $\Gamma(r, \tau, z)$, all of $\Gamma(r, \tau, z)$ except the simplex $H \cap \Gamma(r, \tau, z)$ is contained in the open halfspace bounded by H which contains the origin. If H would meet $(r - \omega\tau^2)\overline{\mathbb{B}}$, then, since $\omega\tau^2 < \delta$, by (4.3), H would meet $b((r^2 - \omega\tau^2)\mathbb{B}) \cap W_1$ at a point not contained in the simplex $H \cap \Gamma(r, \tau, z)$, a contradiction.

Denote by π the projection

$$\pi(x_1, \dots, x_M) = (x_1, \dots, x_{M-1}).$$

Out of simplices building $\Gamma(r, \tau, z)$ choose those which meet W , denote them with $T_j = \Phi_r(S_j)$, $1 \leq j \leq n$ and let C be their union. Since $\tau < \tau_0$ the simplices in $\tau(\mathcal{T} + z)$ contained in U_0 cover U_1 so the simplices S_j , $1 \leq j \leq n$, cover U and consequently $C \cap W$ is a graph over U which cuts W into two connected components. Any path in W connecting points in different components will have to intersect C . What remains of C after we remove the relative interiors of all simplices T_j , $1 \leq j \leq n$, we call the *skeleton* of C and denote by $\text{Skel}(C)$. Obviously $\pi(\text{Skel}(C)) \subset \text{Skel}(\tau(\mathcal{T} + z))$.

For each j , $1 \leq j \leq n$, there is a linear functional ℓ_j on \mathbb{C}^N such that the hyperplane $H_j = \{z \in \mathbb{C}^N: \Re(\ell_j(z)) = 1\}$ contains T_j . We know that each H_j misses $(r - \omega\tau^2)\overline{\mathbb{B}}$.

Since $\Gamma(r, \tau, z)$ is convex it is easy to see that given a sufficiently small $\nu > 0$ the sets

$$\{z \in \mathbb{B}: 1 - \nu < \Re \ell_i(z) < 1\}, \ 1 \leq i \leq n,$$

intersect C in a small neighbourhood $\mathcal{V} \subset C$ of $\text{Skel}(C)$, and the sets $\{z \in \mathbb{B}: \Re \ell_i(z) < 1 - \nu\}$ contain $(r - \omega\tau^2)\overline{\mathbb{B}}$. Note that \mathcal{V} will be arbitrarily small neighbourhood of $\text{Skel}(C)$ provided that $\nu > 0$ is small enough.

Choose ε , $0 < \varepsilon < 1$. Given $L < \infty$ we use a one variable Runge theorem to get a polynomial φ of one variable such that

$$\Re \varphi > L + 1 \text{ on } \{\zeta \in 2\overline{\Delta}: \Re \zeta = 1\}, \quad |\varphi| < \varepsilon/n \text{ on } \{\zeta \in 2\overline{\Delta}: \Re \zeta < 1 - \nu\}.$$

Then $\Phi = \sum_{i=1}^n \varphi \circ \ell_i$ is a polynomial on \mathbb{C}^N such that

$$\Re \Phi > L \text{ on } C \setminus \mathcal{V}, \quad |\Phi| < \varepsilon \text{ on } (r - \omega\tau^2)\overline{\mathbb{B}}.$$

The convex surface $C \subset \text{Sh}((r - \omega\tau^2, r])$, $C \subset \Gamma(r, \tau, z)$ is such that $W \setminus C$ has two components. Let $\mathcal{V}(\eta) \subset C$ be the η -neighbourhood of $\text{Skel}(C)$ where $\eta > 0$ is very small. Write $G = W \cap (C \setminus \mathcal{V}(\eta))$. Then

$$\left. \begin{aligned} (i) \quad & G \subset W((r - \omega\tau^2, r]), \\ (ii) \quad & \text{a path } p: [0, 1] \rightarrow W([r - \omega\tau^2, r]) \setminus G \text{ such that} \\ & |p(0)| = r - \omega\tau^2, \quad |p(1)| = r, \text{ necessarily meets } \mathcal{V}(\eta) \\ (iii) \quad & \text{given } L < \infty \text{ and } \varepsilon > 0 \text{ there is a polynomial } g \text{ on } \mathbb{C}^N \text{ such that} \\ & |g| < \varepsilon \text{ on } (r - \omega\tau^2)\overline{\mathbb{B}} \text{ and } \Re g > L \text{ on } G. \end{aligned} \right\} \quad (4.4)$$

Observe also that π maps $\mathcal{V}(\eta)$ to the η -neighbourhood of $\text{Skel}(\tau(\mathcal{T} + z))$ in \mathbb{R}^{M-1} .

5. Proof of Lemma 3.1, Part 2. Completion of the proof of Lemma 2.1.

Using an easy transversality (or "putting into general position") argument we see that the fact that \mathcal{T} is periodic with respect to Λ implies that there are $q_1, \dots, q_{M-1} \in \mathbb{R}^{M-1}$ such that

$$\text{Skel}(\mathcal{T}) \cap \text{Skel}(\mathcal{T} + q_1) \cap \dots \cap \text{Skel}(\mathcal{T} + q_{M-1}) = \emptyset$$

which implies that there is a $\mu > 0$ such that

$$|x_1 - x_0| + |x_2 - x_1| + \dots + |x_{M-1} - x_{M-2}| \geq \mu$$

whenever $x_i \in \text{Skel}(\mathcal{T} + q_i)$ ($0 \leq i \leq M-1$) where $q_0 = 0$. It then follows that for every $\tau > 0$

$$|x_1 - x_0| + |x_2 - x_1| + \dots + |x_{M-1} - x_{M-2}| \geq \tau\mu \quad (5.1)$$

whenever $x_i \in \text{Skel}(\tau(\mathcal{T} + q_i))$, $0 \leq i \leq M-1$.

Suppose that $1/2 < r_0 < r_M < 1$. Divide the interval $[r_0, r_M]$ into M equal pieces of length $(r_M - r_0)/M$, let $r_0 < r_1 < \dots < r_{M-1} < r_M$ be their endpoints. Choose τ , $0 < \tau < \tau_0$ so small that

$$\tau^2 \omega < \frac{r_M - r_0}{M}. \quad (5.2)$$

Fix a very small $\eta > 0$. For each j , $1 \leq j \leq M$, use $\Gamma(r_j, \tau, q_{j-1})$ to construct $G_j \subset \Gamma(r_j, \tau, q_{j-1})$ as in the preceding section. Then for each j , $1 \leq j \leq M$, we have $G_j \subset W((r_{j-1}, r_j])$ and each path $p: [0, 1] \rightarrow W([r_{j-1}, r_j])$, $|p(0)| = r_{j-1}$, $|p(1)| = r_j$ meets the η -neighbourhood of $\text{Skel}(\Gamma(r_j, \tau, q_{j-1}))$. Thus, if $G = \cup_{j=1}^M G_j$ then for each path $p: [0, 1] \rightarrow W([r_0, r_M]) \setminus G$, such that $|p(0)| = r_0$, $|p(1)| = r_M$, there are t_j , $1 \leq j \leq M$, $0 < t_1 < \dots < t_M < 1$ such that $p(t_j)$ is in the η -neighbourhood of $\text{Skel}(\Gamma(r_j, \tau, q_{j-1}))$, so there is a point $z_j \in \text{Skel}(\Gamma(r_j, \tau, q_{j-1}))$ such that

$$|z_j - p(t_j)| < \eta \quad (1 \leq j \leq M).$$

By (5.1) it follows that

$$\text{length}(p) \geq \sum_{j=1}^{M-1} |p(t_{j+1}) - p(t_j)| \geq \sum_{j=1}^{M-1} |z_{j+1} - z_j| - (M-1)\eta \geq \tau\mu - M\eta.$$

Using (iii) in (4.4) in an induction process again one concludes that given $\varepsilon > 0$ and $L < \infty$ there is a polynomial g on \mathbb{C}^N such that $\Re g > L$ on G and $|g| < \varepsilon$ on $r_0\overline{\mathbb{B}}$.

Thus, if $1/2 < r_0 < r_M < 1$ and if τ , $0 < \tau < \tau_0$ satisfies (5.2) then there is a set $E \subset W((r_0, r_M])$ such that

(i) if $p: [0, 1] \rightarrow W([r_0, r_M]) \setminus E$ is a path such that $|p(0)| = r_0$, $|p(1)| = r_M$ then the length of p exceeds $\tau\mu - M\eta$

(ii) given $L < \infty$ and $\varepsilon > 0$ there is a polynomial g such that $\Re g > L$ on E and $|g| < \varepsilon$ on $r_0\overline{\mathbb{B}}$.

We now prove that for every $A < \infty$ and for every segment $J = (\alpha, \beta]$, $1/2 < \alpha < \beta < 1$ there is a set $E \subset W(J)$ which satisfies (4.1).

So let $1/2 < \alpha < \beta < 1$. Write $(\alpha, \beta] = J_1 \cup J_2 \cup \dots \cup J_\ell$ where $J_1 < J_2 < \dots < J_\ell$ and where $|J_j| = (\beta - \alpha)/\ell$ ($1 \leq j \leq \ell$). For each j we shall construct a set $E_j \subset W(J_j)$ as above. Provided that $0 < \tau < \tau_0$ and $\tau^2\omega < (\beta - \alpha)/(M\ell)$ the set $E = \cup_{j=1}^\ell E_j$ will then have the property that if a path $p: [0, 1] \rightarrow W([\alpha, \beta]) \setminus E$ satisfies $|p(0)| = \alpha$, $|p(1)| = \beta$ then

$$\text{length}(p) \geq \ell(\tau\mu - M\eta) = \ell\tau\mu - M\ell\eta.$$

Suppose that $A < \infty$ is given. We show that it is possible to choose ℓ and τ , $0 < \tau < \tau_0$, so that

$$\tau^2\omega < \frac{\beta - \alpha}{M\ell} \quad (5.3)$$

and

$$\ell\tau\mu = A + 1.$$

In fact, $\tau = (A + 1)/(\ell\mu)$ implies that there is an ℓ_0 such that $0 < \tau < \tau_0$ for every $\ell > \ell_0$. For (5.3) to hold we must have

$$\left(\frac{A + 1}{\ell\mu}\right)^2 \omega < \frac{\beta - \alpha}{M\ell}$$

which is clearly possible provided that $\ell > \ell_0$ is chosen large enough.

So fix such ℓ and such τ and let $\eta > 0$ be so small that $M\ell\eta < 1$. Then

$$\text{length}(p) \geq \ell\tau\mu - M\ell\eta \geq A + 1 - 1 = A.$$

Given $L < \infty$ and $\varepsilon > 0$, an inductive process again produces a polynomial Φ such that $\Re \Phi > L$ on E and $|\Phi| < \varepsilon$ on $\alpha\overline{\mathbb{B}}$. This will complete the proof of Lemma 3.1 and thus the proof of Lemma 2.1 once we have proved that the tessellation \mathcal{T} can be chosen in such a way that, provided that the ball U_0 is small enough, the surfaces $\Gamma(r, \tau, z)$ are convex.

6. Convexity of the surfaces $\Gamma(r, \tau, z)$

We shall now show how to choose a tessellation \mathcal{T} in Section 4 so that after choosing U_0 small enough the polyhedral surfaces $\Gamma(r, \tau, z)$ will be convex. This is the fact that we used in the proof of Lemma 3.1. We essentially follow [G].

Perturb the canonical orthonormal basis in \mathbb{R}^{M-1} a little to get an $(M-1)$ -tuple of vectors e_1, e_2, \dots, e_{M-1} in general position so that the lattice

$$\Lambda = \left\{ \sum_{i=1}^{M-1} n_i e_i : n_i \in \mathbb{Z}, 1 \leq i \leq M-1 \right\} \quad (6.1)$$

will be generic, and, in particular, no more than M points of Λ will lie on the same sphere.

For each point $x \in \Lambda$ there is the *Voronei cell* $V(x)$ consisting of those points of \mathbb{R}^{M-1} that are at least as close to x as to any other $y \in \Lambda$, so

$$V(x) = \{y \in \mathbb{R}^{M-1} : \text{dist}(y, x) \leq \text{dist}(y, z) \text{ for all } z \in \Lambda\}.$$

It is known that the Voronei cells form a tessellation of \mathbb{R}^{M-1} and in our case they are all congruent, of the form $V(0) + x$, $x \in \Lambda$ [CS].

There is a *Delaunay cell* for each point that is a vertex of a Voronei cell. It is the convex polytope that is the convex hull of the points in Λ closest to that point - these points are all on a sphere centered at this point. In our case, when there are no more than M points of Λ on a sphere, Delaunay cells are $(M-1)$ -simplices. Delaunay cells form a tessellation of \mathbb{R}^{M-1} [CS]. In our case it is a *true* Delaunay tessellation, that is, for each cell, the circumsphere of each cell S contains no other points of Λ than the vertices of S . We shall denote by \mathcal{T} the family of all simplices - cells of the Delaunay tessellation for the lattice Λ and this is to be taken as our \mathcal{T} in Section 4. Clearly Λ is precisely the set of vertices of the simplices in \mathcal{T} .

The construction implies that the tessellation \mathcal{T} is periodic with respect to Λ . Thus, there are finitely many simplices S_1, \dots, S_ℓ in \mathcal{T} such that every other simplex of \mathcal{T} is of the form $S_i + w$ where $w \in \Lambda$ and $1 \leq i \leq \ell$. It is then clear by the periodicity that there is an $\eta > 0$ such that for every simplex $S \in \mathcal{T}$ in the η -neighbourhood of the closed ball bounded by the circumsphere of S there are no other points of Λ than the vertices of S .

When we pass from \mathcal{T} to $\tau(\mathcal{T} + z)$ where $\tau > 0$ and $z \in \mathbb{R}^{M-1}$ everything changes proportionally. For instance, for every simplex $S \in \tau(\mathcal{T} + z)$ in the $(\tau\eta)$ -neighbourhood of the closed ball bounded by the circumsphere of S there will be no other vertex of $\tau(\mathcal{T} + z)$ than the vertices of S .

We must now show that if U_0 is chosen sufficiently small then for every r , $1/2 < r < 1$, every $\tau > 0$ and every $z \in \mathbb{R}^{M-1}$ for every simplex $S \in \tau(\mathcal{T} + z)$ contained in U_0 the intersection of the hyperplane H containing $\Psi_r(S)$ with $\Gamma(r, \tau, z)$ is precisely $\Psi_r(S)$.

So let S be such a simplex and let H be the hyperplane in \mathbb{R}^M containing $\Psi_r(S)$. Then H intersects $b(r\mathbb{B})$ in a sphere Γ that is the circumsphere of $\Psi_r(S)$ in H . One component of $b(r\mathbb{B}) \setminus \Gamma$ is contained in the open halfspace bounded by H which contains the origin. Denote this component of $b(r\mathbb{B}) \setminus \Gamma$ by E .

Recall that all vertices of $\Gamma(r, \tau, z)$ are contained in $b(r\mathbb{B})$. Thus, to prove that H contains no other vertex of $\Gamma(r, \tau, z)$ than the vertices of $\Psi_r(S)$ it is enough to show that

$$\text{all vertices of } \Gamma(r, \tau, z) \text{ except the vertices of } \Psi_r(S) \text{ are contained in } E. \quad (6.2)$$

Since $\pi|_{W_0 \cap b(r\mathbb{B})}: W_0 \cap b(r\mathbb{B}) \rightarrow U_0$ is one to one, to show (6.2) it is enough to show that

$$\left. \begin{array}{l} \text{the vertices of all simplices } T \in \tau(\mathcal{T} + z) \text{ contained in } U_0 \\ \text{except the vertices of } S \text{ are contained in the complement} \\ \text{of the bounded domain in } \mathbb{R}^{M-1} \text{ bounded by } \pi(\Gamma). \end{array} \right\} \quad (6.3)$$

Since the $(\tau\eta)$ -neighbourhood of the closed ball in \mathbb{R}^{M-1} bounded by the circumsphere of S contains no other vertex of $\tau(\mathcal{T} + z)$ than the vertices of S it follows that to show (6.3) it is enough to show that

$$\left. \begin{array}{l} \text{provided that } U_0 \text{ is chosen small enough on the outset then for every} \\ S \in \tau(\mathcal{T} + z) \text{ contained in } U_0 \text{ the projection } \pi(\Gamma) \text{ of the circumsphere} \\ \Gamma \text{ of } \Psi_r(S) \text{ in the hyperplane } H \text{ containing } \Psi_r(S) \text{ is contained in the} \\ (\tau\eta)\text{-neighbourhood of the circumsphere of } S \text{ in } \mathbb{R}^{M-1}. \end{array} \right\} \quad (6.4)$$

Given a $(M-1)$ -simplex $T \subset \mathbb{R}^M$ denote by $\Gamma(T)$ the circumsphere of T in the hyperplane containing T . Given a $(M-1)$ -simplex $S \subset \mathbb{R}^{M-1}$ with vertices v_1, \dots, v_M and $\omega > 0$ denote by $\Omega_\omega(S)$ the set of all simplices with vertices $(v_1, q_1), \dots, (v_M, q_M)$ where $q_i \in \mathbb{R}$ satisfy

$$|q_i - q_j| \leq \omega |v_i - v_j| \quad (1 \leq i, j \leq M).$$

PROPOSITION 6.1 *Let $S \subset \mathbb{R}^{M-1}$ be a $(M-1)$ -simplex. Given $\eta > 0$ there is an $\omega > 0$ such that for every $T \in \Omega_\omega(S)$ the projection $\pi(\Gamma(T))$ is contained in the η -neighbourhood of $\Gamma(S)$. Moreover, for any $\tau > 0$ and for any $T \in \Omega_\omega(\tau S)$ the projection $\pi(\Gamma(T))$ is contained in the $(\tau\eta)$ -neighbourhood of $\Gamma(\tau S)$.*

Proof. Let $S \subset \mathbb{R}^{M-1}$ be a simplex with vertices v_1, \dots, v_M and let T be a simplex with vertices $(v_1, q_1), \dots, (v_M, q_M)$. Note that $\pi(\Gamma(T))$ does not change if we translate T in the direction of the last axis so with no loss of generality consider the simplex with the vertices $(v_1, q_1 - q_M), \dots, (v_{M-1}, q_{M-1} - q_M), (v_M, 0)$. We now show that if P is a simplex with vertices $w_1 = (v_1, \beta_1), \dots, w_{M-1} = (v_{M-1}, \beta_{M-1}), w_M = (v_M, 0)$ then

$$\left. \begin{array}{l} \pi(\Gamma(P)) \text{ is arbitrarily close to } \Gamma(S) \text{ provided that } \beta_1, \dots, \beta_{M-1} \\ \text{are sufficiently small.} \end{array} \right\} \quad (6.5)$$

This implies that if $\eta > 0$ then there is an $\varepsilon > 0$ such that if $|q_i - q_M| < \varepsilon$ ($1 \leq i \leq M-1$) then $\pi(\Gamma(T))$ is contained in the η -neighbourhood of $\Gamma(S)$. Picking now $\omega > 0$ so small that $\omega |v_i - v_M| < \varepsilon$ ($1 \leq i \leq M-1$) completes the proof of the first part of proposition. To prove (6.5), let H be the hyperplane containing P and for each j , $1 \leq j \leq M-1$, let H_j be the hyperplane through the midpoint of the segment with endpoints w_j, w_M , perpendicular to $w_M - w_j$. The center C of $\Gamma(P)$ is the intersection of H, H_1, \dots, H_{M-1} . Since these hyperplanes are in general position and change continuously with $\beta_1, \dots, \beta_{M-1}$, the point C and consequently $\Gamma(P)$ changes continuously with $\beta_1, \dots, \beta_{M-1}$. When $\beta_1 = \dots = \beta_{M-1} = 0$ we have $P = S$ so $\Gamma(P) = \pi(\Gamma(P)) = \Gamma(S)$. This implies (6.5).

To prove the second statement of the proposition assume that $\tau > 0$ and that $T \in \Omega_\omega(\tau S)$. Then the vertices of T are $(\tau v_1, p_1), \dots, (\tau v_M, p_M)$ where $|p_i - p_j| \leq \omega|\tau v_i - \tau v_j|$ ($1 \leq i, j \leq M$). Writing $p_i = \tau q_i$ we get that

$$|q_i - q_j| \leq \omega|v_i - v_j| \quad (1 \leq i, j \leq M) \quad (6.6)$$

Thus, the vertices of T are $(\tau v_1, \tau q_1), \dots, (\tau v_M, \tau q_M)$ where (6.6) holds, that is $T = \tau \tilde{S}$ where $\tilde{S} \in \Omega_\omega(S)$. Clearly $\Gamma(T) = \tau \Gamma(\tilde{S})$ and so $\pi(\Gamma(T)) = \pi(\tau \Gamma(\tilde{S})) = \tau \pi(\Gamma(\tilde{S}))$. Since $\tilde{S} \in \Omega_\omega(S)$ the preceding discussion shows that $\pi(\Gamma(\tilde{S}))$ is contained in the η -neighbourhood of $\pi(\Gamma(S))$ so it follows that $\pi(\Gamma(T))$ is contained in the $(\tau\eta)$ -neighbourhood of $\pi(\Gamma(S))$. This completes the proof of Proposition 6.1.

To prove (6.4) recall first that there are finitely many simplices S_1, \dots, S_ℓ in \mathcal{T} such that every other simplex of \mathcal{T} is of the form $S_i + w$ where $w \in \Lambda$ and $1 \leq i \leq \ell$. Thus, there is an $\omega > 0$ such that the statement of Proposition 6.1 holds for every simplex $S \in \mathcal{T} + z$. Recall that $\text{grad} \psi_r$ vanishes at the origin so one can choose U_0 , a ball centered at the origin, so small that

$$|(\text{grad} \psi_r)(x)| < \omega \text{ for all } x \in U_0 \text{ and all } r, \quad 1/2 < r < 1. \quad (6.7)$$

This implies that for every $S \in \tau(\mathcal{T} + z)$ with vertices v_1, v_2, \dots, v_M , contained in U_0 , the simplex $\Psi_r(S)$ with vertices $(v_1, \psi_r(v_1)), \dots, (v_M, \psi_r(v_M))$, by (6.7), satisfies

$$|\psi_r(v_i) - \psi_r(v_j)| \leq \omega|v_i - v_j| \quad (1 \leq i, j \leq M)$$

so the simplex $\Psi_r(S)$ belongs to $\Omega_\omega(S)$ so (6.4) follows by Proposition 6.1. This completes the proof of convexity of surfaces $\Gamma(r, \tau, z)$ and completes the proof of Lemma 3.1. The proof of Lemma 2.1 is complete. Theorem 1.1 is proved.

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